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Cosmological background solutions in theories of mimetic gravity

Candidato: Omar Darwish
Relatore: Prof. Sabino Matarrese

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Chapter 1

Introduction

The nature of the dark matter is one of the greatest puzzles in modern cosmology. In today's standard cosmological model, the $\Lambda - CDM$, dark matter makes up about 80% of all the matter of the universe, with only a small fraction of baryonic matter(protons, neutrons and also electrons in cosmology notation)[1]. From the Coperníncan point of view not only are humans not in a special place in the Universe, but also they are not made of the same stuff that dominates the matter density of the Universe.

Although the big evidence for dark matter no one knows what form it takes. No one knows if dark matter consists of heavy neutrinos, or the Weakly Interacting Massive Particles (WIMPs, the photino, or gravitino or neutralino), or axions(ultra-heavy dark matter particles probably formed at the end of inflation), or a gas of primordial black holes or MACHOs(the unique amongst dark matter candidates in that they have actually been detected [7]) known as Massive Compact Halo Objects.

All this possible candidates act like a specific model of dark matter, the cold dark matter. Cold Dark Matter (CDM) is a model of slow particles (compared to the velocity of light and are slow during Universe's history) that interact very weakly with ordinary matter[2], described by the $\Lambda - CDM$,which assumes the correctness of Einstein's general relativity theory. Dark matter theory fits the data much better than modifications to gravity(with some possible expections such as the one presented here)[3].

There are modifications of Einstein's general relativity theory that explain the anomalous observations in the Universe (such as its accelerated expansion) without introducing undetected sources of matter-energy. One of them is the Mimetic Dark Matter theory, that can explain the phenomenon of the (cold) dark matter, by Chamseddine & Mukhanov [3]. In this model the physical metric of General Relativity is a function of an auxiliary metric and a scalar field appearing through its first derivatives. Applying a general conformal transformation on the auxiliary metric the physical one

[1] https://en.wikipedia.org/wiki/Cold_dark_matter
[2] Interact very weakly with respect to the typical interactions between ordinary matter
[3] http://www.quantumdiaries.org/2015/07/04/why-dark-matter-exists-believing-without-seeing/

remains invariant, so the theory is conformally invariant. The isolated conformal degree of freedom of the physical metric behaves as an irrotational pressureless perfect fluid that can mimic the cold dark matter, with the scalar field playing the role of the velocity potential(of the fluid). Studying the background solutions of it the dark matter appears to be a simple integration constant.

Various studies have dealt with and generalizations of mimetic gravity have been made. For example in [1] mimetic gravity of very general scalar-tensor theories of gravity is obtained via disformal transformations and via Lagrange multiplier. In Chamseddine et al. [4] it is shown that introducing a potential for the scalar field it's possible to mimic gravitational behaviours of normal matter.

In Deruelle & Rua [5] it is elegantly shown that Einstein's General Relativity is invariant under generic disformal transformations of the type $g_{\mu\nu} = F(\Psi, w)l_{\mu\nu} + H(\Psi, w)\partial_\mu\Psi\partial_\nu\Psi$, where $w = l^{\rho\sigma}\partial_\rho\Psi\partial_\sigma\Psi$, F and H two arbitrary functions, $g_{\mu\nu}$ the physical metric and $l_{\mu\nu}$ an auxiliary one. And that there exists a subset of these transformations such that the equations of motion are no longer those of Einstein but the equations of the mimetic dark matter.

This thesis is structured as follows: in chapter one there is a brief review of General Relativity, dealing then with cosmology in chapter two to get a general understanding of how to find some evolution equations, i.e. Friedmann equations, of the Universe. Finally in chapter three, after a presentation of the mimetic gravity theory, are obtained the equations of the evolution of the Universe in the case of Robertson-Walker metric with mimetic dark matter in a Universe without radiation and ordinary matter.

Chapter 2

Gravitation

2.1 Principle of Equivalence

There is a general property for the gravitational field that is true both in classical and relativistic mechanics. All small test bodies in a gravitational field (small means that there isn't a significant modification of the field by the bodies), with the same initial conditions move in the same way, independently of their (intertial) mass. With this property in mind the motion of a body that is studied from a noninertial system of reference can be related to its motion in a gravitational field from an inertial system of reference.

Given a set of free n test bodies in an inertial reference system their motion is uniform and rectilinear, and if all their initial velocities are the same, this will be true for any taken time. And viewed from a noninertial reference system, this test bodies move again with the same velocity during the time. And again, in an inertial reference system but with a gravitational field they all move in the same way.

Thus the properties of the motion in a noninertial system are the same as those in an inertial system in the presence of a specific gravitational field. And any of the two systems can be used to study the test bodies. This is called the principle of equivalence.

The specific gravitational field equivalent to the noninertial system is not completely identical to the gravitational fields generated by a set (discrete or continuous) of energetic objects. There are two differences:

1. a gravitational field, in an inertial system of reference, generated by a set of masses goes to zero at infinite. Contrary to this, a field to which the noninertial reference frame is equivalent increases without limit at infinity, or remains finite in value. The centrifugal force (on a test mass), for example, appearing in a rotating reference system increases without limit moving away from the axis of rotation. Or again, taking a reference frame moving with a non zero constant acceleration, with respect to an inertial frame, studying a test mass from this frame the force that this mass experiences is the same in any given position, and also moving to infinite.

2. a gravitational field 'generated' by a noninertial system vanishes passing to an inertial reference system. A 'real' one, generated by non zero masses, existing also in an inertial frame, cannot be eliminated by any choice of reference system(the mathematical formulation will be seen later).

From this two facts, it is clear that it is impossible, by any choice of reference frame, to eliminate a 'real' field, since it vanishes at infinity. But this can be done in a given region of space, sufficiently small so that the 'real' field can be considered uniform over it. This is can be done by choosing a system in accelerated motion, the acceleration of which is equal to that which would be acquired by a (small) particle placed in the region of the considered field[8].

2.2 The space-time metric

In special relativity in an inertial reference system the interval ds, in cartesian space coordinates, is given by

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

This element has the same value when passing in another inertial reference system(with Lorentz transformations and space-time coordinate shifting).

In a noninertial system of reference the square of an interval appears as a quadratic form of general type in the coordinate differentials[8],

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

, where $g_{\mu\nu}$ are certain functions of the space coordinates x^1, x^2, x^3 and the time coordinate x^0. In a noninertial system, the four-dimensional coordinate system x^0, x^1, x^2, x^3 is curvilinear. And in each curvilinear system of coordinates, $g_{\mu\nu}$ determine all the geometric properties of the system. $g_{\mu\nu}$ is called the space-time metric and $g_{\mu\nu} = g_{\nu\mu}$.

For the principle of equivalence non-inertial reference frames are equivalent to certain gravitational fields given in an inertial reference frame. Those fields are so related to $g_{\mu\nu}$. The same thing is for the real gravitational fields, that the field is the modification of the metric of the space-time and so is determined by the values of $g_{\mu\nu}$. So, the geometrical properties of the space-time, the metric, are determined by physical phenomena and are not invariable properties of the space and the time[8]. The theory of the space-time and the one for gravitation are the same theory.

In an inertial reference system, when using cartesian space coordinates $x^{1,2,3} = x, y, z$ and the time $x^0 = ct$, $g_{\mu\nu}$ is given by $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$, $g_{\mu\nu} = 0, \mu \neq \nu$.

A four-dimensional system of coordinates with these values of $g_{\mu\nu}$ is called galilean. The galilean metric is denoted by $g_{\mu\nu}^{(0)} = diag(1,-1,-1,-1) = \eta_{\mu\nu}$.

By an appropriate choice of coordinates the quantities $g_{\mu\nu}$ can be brought to the galileain form at any individual point of the non-galileian space-time: reducing to diagonal form the $g_{\mu\nu}$. Such coordinate system is said to be galilean for the given point. After the reduction to diagonal form at a given point, the matrix of the quantities $g_{\mu\nu}$ has one positive(negative) and three negative (positive) principal values. From this it follows, in particular, that for continuity with the galileian metric, the determinant g , formed from the quantities $g_{\mu\nu}$, is always negative for a real space-time.

For a change of coordinate system x to another one $\tilde{x}$ the components $g_{\mu\nu}$ vary in this way (as a 2-tensor):

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x)$$

The invariant volume element under coordinate transformations is : $\sqrt{-g}d\Omega$, $g = det(g_{\mu\nu})$ and $d\Omega = dx^0 dx^1 dx^2 dx^3 = d^4x$.

If the transformation of $g_{\mu\nu}$ to the galileian form is possible for all points of the space the space-time is flat. And the geometry of the space is euclidean. If this global transformation is not possible, there is a curved space-time. So, with real gravitational fields there is a curved space-time. For a gravitational field, the real one in which the space-time is curved or the ones generated by a non-inertial reference system in which the space-time is flat, the geometry of the space is non-euclidean.

2.3 Principle of General Covariance

In general relativity a reference system is determined by an infinite quantity of bodies that fill the space as a continuum medium in which to each body is attached a clock that indicates an arbitrary time describing uniquely the points of the four-dimensional space-time [8].

Because the choice of a reference system, and coordinate system, is arbitrary, nature laws much be expressed in a covariant way for each four-dimensional coordinate system. There isn't any special frame with respect to others. The mathematical form to write this is that physics laws must be expressed with tensors:

$$T^{\mu_1\mu_2\ldots\mu_p}_{\nu_1\nu_2\ldots\nu_q} = 0$$

If this is true for a given coordinate system it will be also for any another one(with tensors transformation rules).

In order to write the equations that govern physical phenomena in a gravitational field, the Principle of Equivalence tells that the laws of physics in a gravitational field must reduce to those of Special Relativity in the galilean coordinate system for a given point and, as said, maintain its form under a general coordinate transformation. This is the principle of general covariance.

There is a simple heuristic principle that permits to write the equations of physics in any reference system in general relativity: if an equation is written for an arbitrary metric g and connection ∇, when restricted to Minkowski spacetime it still holds with $g_{\mu\nu}$ replaced by $\eta_{\mu\nu}$ and ∇_μ replaced by ∂_μ. Conversely, given any special relativistic equation written in tensorial form, the generally covariant equation is obtained replacing $\eta_{\mu\nu}$ by $g_{\mu\nu}$ and ∂_μ by ∇_μ and the volume element d^4x by $\sqrt{|g|}d^4x$. This resulting equation will automatically satisfy the Principle of General Covariance and the Equivalence Principle. This procedure is called 'minimal coupling'. The fact that the generalized equation from flat to a curved space-time describes gravity it's not trivial [2].

2.4 Time Intervals

It was said that in general relativity the choice of coordinates is arbitrary. But how it is possible from their values find real time intervals? Considering two infinitesimally near events, that happen in the same point of the space. Then $ds^2 = c^2 d\tau^2$ with τ the real interval time (or proper time) measured by an observer in the point of the space in a local galileian system, $dx^i = 0, i = 1, 2, 3$. Putting it in the general expression of ds^2, $(c = 1)$,$ds^2 = d\tau^2 = g_{00}(dx^0)^2$. So,

$$d\tau = \sqrt{g_{00}}dx^0$$

and the real time interval is

$$\tau = \int \sqrt{g_{00}}dx^0$$

From this the condition $g_{00} > 0$, otherwise the reference system can't be realized by real bodies. But it's always possible to choose a reference system in which such that $g_{00} > 0$ [8].

To synchronize the clocks indications of these two near events the relation $\Delta x^0 = -\frac{g_{0i}dx^i}{g_{00}} = 0$ must be satisfied [8].

A unique synchronization in all the space is possible if all g_{0i} are zero. And for any gravitational field it is always possible to find a reference system with this property, or in which clocks synchronization is always possible[8].

If $g_{00} = 1$ then $x^0 = t$ is the proper time in each point of the space. A reference system with

$$g_{00} = 1, g_{0i} = 0$$

is called synchronous.

2.5 Curvature

Given in a specific coordinate system in the curved space-time a four-vector A^μ in the point x^μ, after the operation of parallel transport in the point $x^\mu + dx^\mu$, the difference between A^μ and $A^\mu + dA^\mu$(that is A^μ calculated in $x^\mu + dx^\mu$) in $x^\mu + dx^\mu$ is $DA^\mu = dA^\mu - \delta A^\mu = dA^\mu + \Gamma^\mu_{\nu\rho} A^\nu dx^\rho$. $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols with the property $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$.

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \left(\frac{\partial g_{\sigma\nu}}{\partial x^\rho} + \frac{\partial g_{\sigma\rho}}{\partial x^\nu} - \frac{\partial g_{\nu\rho}}{\partial x^\sigma} \right)$$

The definition of covariant derivative is $DA^\mu = A^\mu_{;\nu} dx^\nu = \nabla^\mu_\nu A dx^\nu$. And for the covariant vector $A_\mu = g_{\mu\nu} A^\nu$, $DA_\mu = A_{\mu;\nu} dx^\nu$.

It can be shown [8] that

$$A_{\mu;\nu;\rho} - A_{\mu;\rho;\nu} = A_\sigma R^\sigma_{\mu\nu\rho}$$

where $R^\sigma_{\mu\nu\rho}$ is a four order tensor, the Riemann tensor defined by

$$R^\sigma_{\mu\nu\rho} = \frac{\partial \Gamma^\sigma_{\mu\rho}}{\partial x^\nu} - \frac{\partial \Gamma^\sigma_{\mu\nu}}{\partial x^\rho} + \Gamma^\sigma_{\eta\nu} \Gamma^\eta_{\mu\rho} - \Gamma^\sigma_{\eta\rho} \Gamma^\eta_{\mu\nu}$$

The Ricci tensor is $R_{\mu\nu} = g^{\sigma\rho} R_{\sigma\mu\rho\nu}$, $R_{\mu\nu} = R_{\nu\mu}$. The contraction of $R_{\mu\nu}$, $R = g^{\mu\nu} R_{\mu\nu}$ is the scalar curvature of the space.

In the presence of a flat space-time the curvature tensor is zero. This follows from the fact [8] that in a flat space time it's always possible to choose a coordinate system such that the Christoffel symbols are zero everywhere and so $R^\sigma_{\mu\nu\rho} = 0$. Then from the tensor nature of the curvature tensor the last equation is true for any other coordinate system on the flat space-time. The inverse is also true , if $R^\sigma_{\mu\nu\rho} = 0$ everywhere in a given coordinate system, then the space-time taken in consideration is flat[8].

In a curved space-time it's always possible to choose a coordinate system such that the Christoffel symbols are zero only locally in a given point (this is the mathematical expression of the elimination of a real field in the point of the space expressed in 2.1). But the curvature tensor is not zero in that point (Christoffel symbols have non zero derivatives in the point).

2.6 Motion in a gravitational field

As in relativistic mechanics the motion for a free body particle in a gravitational field is given by the principle of least action(the mass m of the particle is such that it doesn't influence the gravitational field):

$$\delta S = -mc\delta \int ds = 0$$

, between two fixed points of the space time. The equations of motion derived are (for example by the principle of least action or by minimal coupling):

$$\frac{d^2x^\rho}{ds^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$$

, where $\frac{d^2x^\rho}{ds^2}$ is the four-acceleration of the particle and

$$-m\Gamma^\rho_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$

the 'four-force' on the particle in the gravitational field in the given reference system. With an adeguate reference choice it is always possible to set to zero all the $\Gamma^\rho_{\mu\nu}$ in a given point of the space-time(see 2.5). From this

$$\frac{d^2x^\rho}{ds^2} = 0$$

the relativistic mechanic expression for a free body particle. The gravitational field in the infinitesimal volume about the point is zero. This is the possibility to choose a localy inertial frame around a point of the space-time, expression of the principle of equivalence.

2.7 Equations of the gravitational field

Now is the moment to see the connection between matter and gravity. The Einstein equations explain how matter generates the gravitational field and then the field modificate the motion of the matter. Solving these equations and knowing the state equation that relates pressure with the density of matter(or energy), one can obtain the tensor $g_{\mu\nu}$, the matter distribution and its motion(that are described in the stress-energy tensor $T_{\mu\nu}$). Here the derivation of Einstein equations follows the one presented in [8], through the principle of least action.

The total action of the gravitational field is $S = S_g + S_m$, where S_g is the Einstein-Hilbert action and S_m the action for matter. The equations of the field are obtained by the principle of least action(that says that the action S is stationary under the variation

$\delta S = 0$, it is supposed that the variations of the field are zero on the boundary of the system) :

$$\delta S = \delta S_g + \delta S_m = 0$$

The gravitational action has the form $S_g = -\frac{1}{2}\int R\sqrt{-g}d\Omega$ (see [8] for why this form), the variation

$$\delta S_g = -\frac{1}{2}\delta\int R\sqrt{-g}d\Omega = \delta\int g^{\mu\nu}R_{\mu\nu}\sqrt{-g}d\Omega =$$

$$= \int \delta(g^{\mu\nu}R_{\mu\nu}\sqrt{-g})d\Omega$$

$$= -\frac{1}{2}\int(R_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu} + R_{\mu\nu}g^{\mu\nu}\delta\sqrt{-g} + g^{\mu\nu}\sqrt{-g}\delta R_{\mu\nu})d\Omega$$

, where $g = det(g_{\mu\nu})$,

$$\sqrt{-g}d\Omega = \sqrt{-g}dx^0dx^1dx^2dx^3 = \sqrt{-g}d^4x$$

is the invariant volume element, R the Ricci scalar and $R_{\mu\nu}$ is the curvature tensor. The integral is done in the 4-volume between the hypersurfaces of the space-time with $x^0_A < x^0 < x^0_B$.

From a corollary of Jacobi's formula[1] $log(g) = Tr(log(g_{\mu\nu})))$, where Tr is the trace of a matrix, and $B = log(A)$ is the inverse of the exponential of a matrix A, i.e $e^B = A$.

Then

$$\sqrt{-g} = \sqrt{-e^{Tr(log(g_{\mu\nu}))}} = \sqrt{-e^{-Tr(log(g^{\mu\nu}))}} \to \delta\sqrt{-g} = \delta(\sqrt{-e^{-Tr(log(g^{\mu\nu}))}})$$

$$= \delta(-e^{-\frac{1}{2}Tr(log(g^{\mu\nu}))}) =$$

$$= e^{-\frac{1}{2}Tr(log(g^{\mu\nu}))}\delta(-\frac{1}{2}Tr(log(g^{\mu\nu}))) = -\frac{1}{2}\sqrt{-g}\delta g^{-1}Tr((log(g^{\mu\nu}))) =$$

$$= -\frac{1}{2}\sqrt{-g}Tr(g_{\mu\nu}\delta g^{\nu\rho}) = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$$

[1] https://en.wikipedia.org/wiki/Logarithm_of_a_matrix

because the trace is a linear operator.

Another way is because $g_{\mu\nu}$ is an invertible matrix, from Jacobi's formula [2] $d(det(g_{\mu\nu})) = dg = gTr(g^{\mu\rho}dg_{\rho\nu})$, and so in terms of the variation $\delta g = g(g^{\mu\nu}\delta g_{\mu\nu})$. And

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2\sqrt{-g}}gg^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$$

And so

$$\delta\int R\sqrt{-g}d\Omega = \int(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu}\sqrt{-g}d\Omega + \int g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}d\Omega$$

For

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\delta(\frac{\partial\Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial\Gamma^{\rho}_{\mu\rho}}{\partial x^{\nu}} + \Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\nu\sigma})$$

Even if $\Gamma^{\rho}_{\mu\nu}$ is not a tensor it can be shown that $\delta\Gamma^{\rho}_{\mu\nu}$ is a tensor. So it is possible to calculate the variation $\delta R_{\mu\nu}$ in a locally inertial reference system(where all $\Gamma^{\rho}_{\mu\nu}$ are zero). This gives

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\delta(\frac{\partial\Gamma^{\rho}_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial\Gamma^{\rho}_{\mu\rho}}{\partial x^{\nu}}) = g^{\mu\nu}\frac{\partial}{\partial x^{\rho}}\delta\Gamma^{\rho}_{\mu\nu} - g^{\mu\rho}\frac{\partial}{\partial x^{\rho}}\delta\Gamma^{\nu}_{\mu\nu} = \frac{\partial w^{\rho}}{\partial d^{x}\rho}$$

where

$$w^{\rho} = g^{\mu\nu}\delta\Gamma^{\rho}_{\mu\nu} - g^{\mu\rho}\delta\Gamma^{\nu}_{\mu\nu}$$

is a vector. In any reference system

$$g^{\mu\nu}R_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\rho}}(\sqrt{-g}w^{\rho})$$

This gives:

$$\int g^{\mu\nu}\delta R_{\mu\nu}\sqrt{-g}d\Omega = \int\frac{\sqrt{-g}w^{\rho}}{\partial x^{\rho}}d\Omega$$

and for the Guass theorem this can be transformed in an integral over the hypersurface that encloses all the four-volume over which the integral is made. But this term vanishes because the fields variation on the boundaries are zero.

In the end,

$$\delta S_g = -\frac{1}{2}\int(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu}\sqrt{-g}d\Omega$$

On the other hand,

$$S_m = -\frac{1}{2}\int L_m\sqrt{-g}d\Omega$$

[2]https://en.wikipedia.org/wiki/Jacobi's_formula

, where L_m is the density of Lagrangian for the matter.

$$\delta S_m = -\frac{1}{2}\delta \int L_m \sqrt{-g} d\Omega = \delta S_m = -\frac{1}{2} \int T_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g} d\Omega$$

, where $T_{\mu\nu}$ is the stress-energy tensor for matter(with the electromagnetic field).

Now, using the principle of least action

$$\delta S = \delta S_g + \delta S_m = -\frac{1}{2} \int (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - T_{\mu\nu}) \delta g^{\mu\nu} \sqrt{-g} d\Omega = 0$$

for each arbitrary $\delta g^{\mu\nu}$(that is zero on the boundary). From this the equations of the gravitational field or Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$$

The tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

is the Einstein tensor. So Einstein equations can be written also as

$$G_{\mu\nu} = T_{\mu\nu}$$

The stress-energy tensor satisfy the local conservation of energy and momentum, the continuity equation

$$\nabla_\nu T^{\mu\nu} = 0$$

Chapter 3

Cosmology

3.1 Cosmological Principle

The basic assumption in this work, the cosmological principle, is that the Universe on large scales is homogenous, it looks the same in each point, and isotropic, it looks the same in all directions. Large scales means that the distances in role are much larger than those between galaxies. If a distribution(matter, radiation,...) is isotropic about every point, then that does enforce homogeneity as well [7].

The density of the distribution is the average density with respect to the distances such that the cosmolgical principle is true. It will be supposed the same everywhere. For the cosmological principle the universe is imaged to be filled by, perfect fluids. A fluid because seen from large scales the universe appears filled by a continuum of matter, with the average density described above. Perfect because any non-zero viscosity would destroy isotropy. Homogenous, perfect fluids for example are dust($p = 0$), dark matter and radiation. Given a reference system any, perfect, fluid in the universe is described by the stress-energy tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$$

, ρ and p are functions of time (for the cosmological principle functions depend only on time) describing the density and the pressure, with a four-velocity vector that follows $u_\mu u^\mu = 1$ (signature$(g_{\mu\nu}) = (+1, -1, -1, -1)$).

3.2 Components of the universe , expansion of the universe

After defined the stress energy tensor of the fluid model described above, there's the need of an equation of state, i.e. $p = w(\rho)$. In cosmology perfect fluids are usually described by $p = w\rho$ where w is some constant that is different for each type of perfect fluid taken in consideration.

Astronomical data of the distribution of galaxies in the space and the isotropy of the cosmic microwave background don't contradict the assumptions of the cosmological principle.

The universe is filled with a mixture of different matter components[1].

- Matter, all form of matter for which the pressure is less smaller than the density. The case of a gas of non relativistic particles. This include:
  - Baryons: for cosmologists is the ordinary matter, nuclei and electrons
  - Dark Matter: most of the matter in the universe is in the form of invisible dark matter. Usually thought to be a new heavy particle species, its nature is not known
- Radiation, anything whose pressure is about one third of the density. The case of a gas of relativistic particles. This include:
  - Photons: the early universe was dominated by photons. Being massless, they are always relativistic. Today, are detected in the form of the cosmic microwave background.
  - Neutrinos: are extremely weakly interacting particles. There is a significant experimental evidence that they possess a non-zero rest mass. For now they can be treated as massless.
- Dark Energy, matter and radiation aren't enough to describe the evolution of the universe. Instead, the universe today seems to be dominated by a mysterious negative pressure component $p = -\rho$.

An observational evidence in cosmology is that almost eveything in the Universe appears to be moving far away form Earth(taken as a point, or any other point in the Universe), and the further away something is, the more rapid its recession appears to be. This is summariezed by Hubble's law. These velocities are measured via redshift(Doppler effect for light waves).

3.3 Metric

Homoegeneity and isotropy of the space imply the choice of a universal time such that for each instant the metric of the space is the same in all its points and in all directions. Indeed, if time depended on the space coordinates of the point of measurements, then this can be distinguished from other points. This contradicts the cosmological principle.

[1] http://www.damtp.cam.ac.uk/user/db275/Cosmology/Lectures.pdf

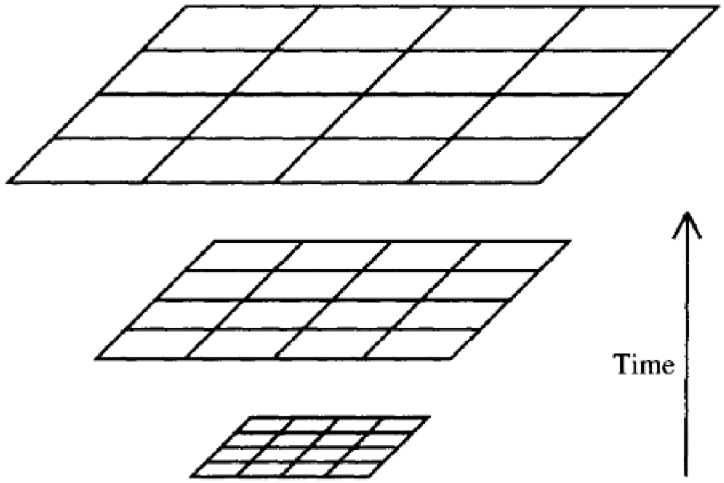


Figure 3.2 The comoving coordinate system is carried along with the expansion, so that any objects remain at fixed coordinate values.

Figure 3.1: Comoving Coordinates

One have now to choose a reference system to study the system. As said general relativity allows to take any one. It is convenient to choose the one in motion in each point of the space with the fluid. The fluid is the reference system, in which each its infinitesimal body component has a clock that indicate a time(it was said that there is only a universal time, later it will be shown better how the various clocks can share a unique time, synchronization). The observer of this reference system is called comoving observer. Any other observer would see an apparent break of the isotropy for the orientation of the velocities of the fluid in its different points.[2]

For a comoving observer pressure p of the fluid can be assumed to be everywhere zero(or at least constant), so that $w = 0$. In the universe there aren't pressure gradients, because the density and the pressure are everywhere the same. Pressure does not contribute a force helping the expansion along. It's effect is solely through the work done as the Universe expands. It's important to note that in the scales in role the internal structure of the bodies in the universe are not taken in consideration(don't care about internal p of the bodies). It is also assumed that the radiation contained in the space has energy density and pressure much smaller than the energy density of the fluid [7].

After taking the comoving reference system it's important to find the metric of the space-time. Later equations of motion will be solved. For the equivalence of the metric in all directions its components $g_{0i} = g_{i0} = 0$ (otherwise no equivalence of the different directions about a given point). This condition allows the synchronization

[2]Note that in this reference system the fluid has to be also irrotational

of the clocks in different points of the space. So,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{00}dx_0{}^2 + 2g_{i0}dx^0dx^i + g_{ii}dx^idx^i = g_{00}dt^2 - dl^2$$

, where

$$dl^2 = \gamma_{ij}dx^idx^j = -g_{ij}dx^idx^j$$

.($c = 1$) For what said above g_{00} is a function only of the time. Choosing $g_{00} = 1$, t is the synchronous proper time in each point of the space(this is the universal time, see 2.4).

3.4 Robertson-Walker metric

For the comological principle the spatial part of the metric, at a given (universal) time, has constant curvature. It can be shown that the most general spatial metric with this property is

$$dl^2 = \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)$$

, in spherical polar coordinates. $k >,=,< 0$, respectively for, spherical, flat and hyperbolic spatial geometries. Incorporating it into a space-time metric. The only further dependence that can be put is a time one. The space is allowed to grow or shrink with time. This leads to Roberston-Walker metric (spherical space coordinates)

$$ds^2 = dt^2 - a^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)]$$

, where $a(t)$ is the scale factor of the Universe, a quantity that measures the universal expansion rate.

It will be supposed $k = 0$ and ds^2 will be simply $ds^2 = dt^2 - a^2(t)\delta_{ij}dx^idx^j$(with cartesian space coordinates). It is useful to introduce, by a conformal transformation, the conformal time η by

$$dt = ad\eta$$

Then the interval element is written as $ds^2 = a^2(\eta)(d\eta^2 - \delta_{ij}dx^idx^j)$.

3.5 Friedmann equations

The equations that describe the evolution of the universe are the Friedmann equations: these are differential equations in the universal time and includes the scale factor of the universe $a(t)$, the density $\rho(t)$ and the pressure of the universe $p(t)$(because the evolution of the Universe, governed by general relativity, is determined not only by geometry but also by its content). This functions can be expressed also in function of the conformal time η.

The Friedmann equations can be derived from Einstein equations and substituting directly in them the values of R, $R_{\mu\nu}$ and $T_{\mu\nu}$ for a isotropical, uniform and flat universe. Or directly from the Einstein-Hilbert Action supposing to work in the Robertson-Walker metric with conformal time(or also proper time).

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}(R(g) + L_m)$$

, R Ricci scalar and L_m the density of Lagrangian for matter.

The second way will be shown later, so it can be useful to see with the first one.

Einstein equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}$$

The values of the metric tensor are(coordinate system t, x, y, z): $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -a^2$(the other values are zero).

In Robertson-Walker metric with proper time[3] :

$$R_{00} = -3\frac{\ddot{a}}{a}$$

$$R_{ij} = -(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2})g_{ij}$$

$$R = -\frac{6}{a^2}(a\ddot{a} + \dot{a}^2)$$

The Einstein tensor has the following components

$$G_{00} = 3\frac{\dot{a}^2}{a^2}$$

$$G_{0i} = 0$$

[3]http://www.blau.itp.unibe.ch/newlecturesGR.pdf

$$G_{ij} = (\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})g_{ij}$$

As said a perfect (irrotational) fluid has the stress-energy tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$$

with $u^\mu = (1, 0, 0, 0)$ in the comoving coordinate system(in which the fluid is at rest). It is supposed also to have a barytropic fluid, in which the pressure depends only on density, $p = p(\rho)$. A good model for a cosmological fluid arise from considering linear this relationship, i.e $p = w\rho$, w is the state parameter.

Common cases are:

1. Non interacting particles, $p = 0$ and so $w = 0$. This is the matter dust.
2. Electromagnetic radiation, $w = \frac{1}{3}$, so the radiation has equation of state $p = \frac{\rho}{3}$.

Because of isotropy there are only two independent equations for the Einstein equations, the 00-component and one of the non-zero ij components.

From the time Einstein equation:

$$(\frac{\dot{a}}{a})^2 = \frac{\rho}{3}$$

From the spatial Einstein equations:

$$2\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 = -p$$

From these two equations, or by the continuity equation for $T_{\mu\nu}$, it can be derived the energy continuity equation

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a}$$

As an example, two simple cases for Friedmann Equations solutions:

- Universe dominated by Dust:from continuity equation $\frac{d(\rho a^3)}{dt} = 0$, then $\rho \propto a^{-3}$. If t_0 is the present time, it can be fixed $a(t_0) = 1$ and so write

$$\rho = \frac{\rho_0}{a^3}$$

substituting into the first Friedmann equation

$$\dot{a}^2 = \frac{\rho_0}{3}\frac{1}{a}$$

The full solution is therefore

$$a(t) = (\frac{t}{t_0})^{2/3}$$

and

$$\rho(t) = \frac{\rho_0 t_0^2}{t^2}$$

- Universe dominated by Radiation:$p = \rho/3$, from continuity equation $\frac{d(\rho a^4)}{dt} = 0$, or $\rho \propto a^{-4}$. It follows that

$$a(t) = (\frac{t}{t_0})^{1/2}$$

and

$$\rho(t) = \frac{\rho_0 t_0^2}{t^2}$$

- There is also the case where there is a mixture of both matter and radiation. And now $\rho = \rho_{mat} + \rho_{rad}$. The equations of the study of the expansion are not presented, but the figure shows the evolution of the Universe containing matter and radiation, with the radiation initially dominating. Eventually the matter comes to dominate, and as it does so the expansion rate speeds up from $a(t) \propto t^{1/2}$ to the $a(t) \propto t^{2/3}$law.

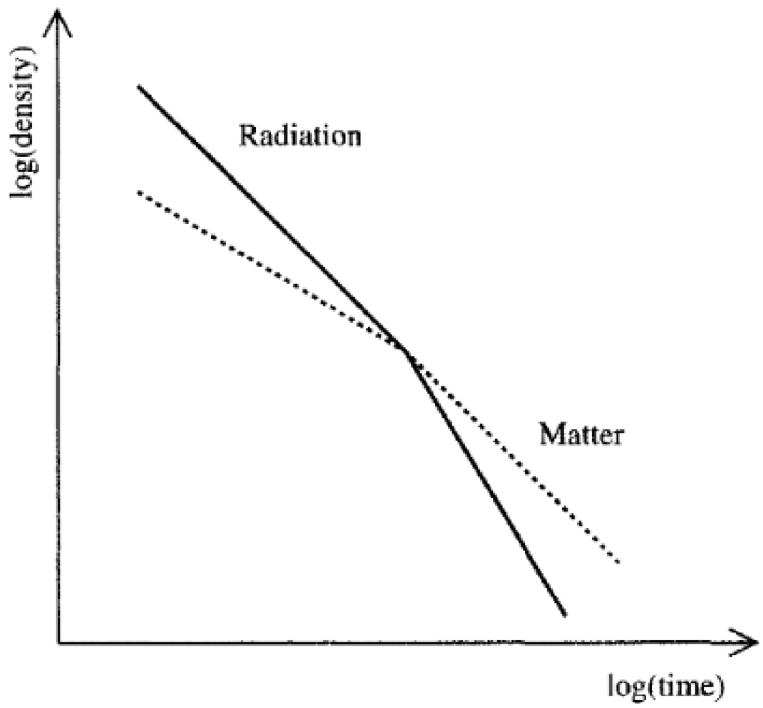


Figure 5.2 A schematic illustration of the evolution of a Universe containing radiation and matter. Once matter comes to dominate the expansion rate speeds up, so the densities fall more quickly with time.

Figure 3.2: Radiation-Matter domination

Chapter 4

Mimetic Dark Matter

4.1 Mimetic Gravity via disformal transformations

In [5] it was shown that Einstein's equations are generically invariant under the redefinition of the metric $g_{\mu\nu}$by means of disformal transformations. Here are presented only the main steps to show this. See [5] for details.

Taken $g_{\mu\nu}$, this can be written as a function of a metric $l_{\mu\nu}$ and a scalar field Ψ with a generic disformal transformation, i.e. :

$$g_{\mu\nu} = F(\Psi, w) l_{\mu\nu} + H(\Psi, w) \partial_\mu \Psi \partial_\nu \Psi$$

where $w = l^{\rho\sigma} \partial_\rho \Psi \partial_\sigma \Psi$ and the functions F and H are a priori arbitrary.

To find the equations of motion of gravity from the Einstein-Hilbert action,

$$S = -\frac{1}{2} \int (R\sqrt{-g} + L_m) = S_g + S_m$$

As usual, the variation of the action with respect to $g_{\mu\nu}$ is taken,

$$\delta S = -\frac{1}{2} \int d^4x \sqrt{-g} (G^{\mu\nu} - T^{\mu\nu}) \delta g_{\mu\nu}$$

, $T^{\alpha\beta} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\alpha\beta}}$.

Writing $\delta g_{\mu\nu}$ as a function of $l_{\mu\nu}$ and Ψ and their variations, and then taking the variation of the action one time with respect to $l_{\mu\nu}$ and the other with respect to Ψ two equations of motion are obtained. The one derived by the variation of $l_{\mu\nu}$, contracted with respect to $l_{\mu\nu}$ and $\partial_\mu \Psi \partial_\nu \Psi$ yield to the system (1)

$$A(F - w\frac{\partial F}{\partial w}) - Bw\frac{\partial H}{\partial w} = 0$$

$$Aw^2\frac{\partial F}{\partial w} - B(F - w^2\frac{\partial H}{\partial w}) = 0$$

where $A = (G^{\rho\sigma} - T^{\rho\sigma})l_{\rho\sigma}$ $B = (G^{\rho\sigma} - T^{\rho\sigma})\partial_\rho\Psi\partial_\sigma\Psi$

A system of two homogenous equations for the two unkowns A and B. The determinant det of the system is

$$\det = w^2F\tfrac{\partial}{\partial w}(H + \tfrac{F}{w})$$

In the generic case when it is not zero the only solution of the system above is $A = B = 0$ and the equations of motion reduce to the standard Einstein Equations of General Relativity $G_{\mu\nu} = T_{\mu\nu}$ for the metric $g_{\mu\nu}$

$$F(G^{\mu\nu} - T^{\mu\nu}) = 0$$

$$F \neq 0$$

Then extremizations of the Einstein-Hilbert action with respect to the 'disformed' metric $g_{\mu\nu}$ or with respect to $l_{\mu\nu}$ and Ψ are equivalent, yielding to standard Einstein's equations.

But when $\det = 0$, with $F \neq 0$, then $\frac{\partial}{\partial w}(H + \frac{F}{w}) = 0$ and the function $H(w, \Psi)$ takes the form

$$H(w, \Psi) = -\frac{F(w, \Psi)}{w} + h(\Psi) \tag{4.1}$$

The solution of the system (1) yield to $B = wA$. Supposing $h \neq 0$ and defining Φ such that $\frac{d\Phi}{d\Psi} = \sqrt{|h|}$ one gets [5],

$$G_{\mu\nu} - T_{\mu\nu} = (G - T)\partial_\mu\Phi\partial_\nu\Phi$$

$$2\nabla_\rho[(G - T)\partial^\rho\Phi] = 0$$

, supposing $g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi = 1$. These are not Einstein's equations, because includes also the extra term $(G - T)\partial_\mu\Phi\partial_\nu\Phi$ that in general is not zero that gives novel solutions.

The case of $F = w$ and $H = 0$ is the same of mimetic gravity, that is made by a conformal transformation(a type of the disformal ones), which is described in the next section in which also the meaning of $G_{\mu\nu} - T_{\mu\nu} = (G - T)\partial_\mu\Phi\partial_\nu\Phi$ will be discussed.

4.2 Mimetic Gravity

Here it is presented a formulation, by Chamseddine & Mukhanov [3], in which the equations of motion of the system are similar to Einstein's equations of motion but with an extra term that mimics cold dark matter even in absence of 'standard' matter.

The Einstein-Hilbert action is

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}[R + L_m]$$

, with L_m the lagrangian density for matter and $8\pi G = 1$, $c = 1$.

The physical metric $g_{\mu\nu}$ is represented as a function of $\tilde{g}_{\mu\nu}$ an auxiliary metric and a scalar field ϕ.

$$g_{\mu\nu} = (\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi)\tilde{g}_{\mu\nu} = P(x)\tilde{g}_{\mu\nu}$$

, $x = x^\alpha$ the coordinates of the point of space-time taken in consideration, ∂_α partial derivative with respect to x^α (the reference system is not changed after this transformation).

Then the action is constructed with the physical metric $g_{\mu\nu}$ considered as a function of the scalar field ϕ and the auxiliary metric $\tilde{g}_{\mu\nu}$:

$$S = -\frac{1}{2}\int d^4x\sqrt{-g(\tilde{g}_{\mu\nu},\phi)}[R(g_{\mu\nu}(\tilde{g}_{\mu\nu},\phi)) + L_m]$$

In this case the metric $g_{\mu\nu}$ has its conformal mode isolated in the scalr field ϕ, and now the physical metric is invariant with respect to the conformal transformation of the auxiliary metric $\tilde{g}_{\mu\nu}$,

$$g_{\mu\nu} \to g_{\mu\nu}$$

when

$$\tilde{g}_{\mu\nu} \to \Omega^2\tilde{g}_{\mu\nu}$$

($\tilde{g}^{\mu\nu} \to \Omega^{-2}\tilde{g}^{\mu\nu}$). And from $g_{\mu\nu} = P(x)\tilde{g}_{\mu\nu}$ the isolation of the conformal mode of the metric is seen in

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1$$

S is invariant under the conformal transformation of $\tilde{g}_{\mu\nu}$ above because it depends only on $g_{\mu\nu}$ which is conformally invariant by itself.

Taking the variation of the action with respect to the physical metric

$$\delta S = \int d^4x\frac{\delta S}{\delta g_{\alpha\beta}}\delta g_{\alpha\beta} = -\frac{1}{2}\int d^4x\sqrt{-g}(G^{\alpha\beta} - T^{\alpha\beta})\delta g_{\alpha\beta}$$

, $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}$ is the Einstein tensor and $T^{\mu\nu}$ is the energy momentun tensor for the matter.

The variation $\delta g_{\alpha\beta}$ can be expressed in terms of the variations $\delta\tilde{g}_{\alpha\beta}$ and $\delta\phi$, taking the form

$$\delta g_{\alpha\beta} = \delta(P\tilde{g}_{\alpha\beta}) = P\delta\tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta}\delta P =$$

$$= P\delta\tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta}\delta(\tilde{g}^{\kappa\lambda}\partial_\kappa\phi\partial_\lambda\phi)$$

$$= P\delta\tilde{g}_{\alpha\beta} + \tilde{g}_{\alpha\beta}(-\tilde{g}^{\kappa\mu}\tilde{g}^{\lambda\nu}\delta\tilde{g}_{\mu\nu}\partial_\kappa\phi\partial_\lambda\phi + 2\tilde{g}^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi) =$$

$$= P\delta\tilde{g}_{\mu\nu}(\delta^\mu_\alpha\delta^\nu_\beta - g_{\alpha\beta}g^{\kappa\mu}g^{\lambda\nu}\partial_\kappa\phi\partial_\lambda\phi) + 2g_{\alpha\beta}g^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi$$

because $\delta g^{\mu\nu} = -g^{\mu\alpha}(\delta g_{\alpha\beta})g^{\beta\nu}$ and the derivatives of ϕ, and the metric tensor also, are simmetric.

Thus the variation of the actions is written as:

$$\delta S = -\frac{1}{2}\int d^4x\sqrt{-g}(G^{\alpha\beta} - T^{\alpha\beta})\delta g_{\alpha\beta} =$$

$$= -\frac{1}{2}\int d^4x\sqrt{-g}(G^{\alpha\beta} - T^{\alpha\beta})(P\delta\tilde{g}_{\mu\nu}(\delta^\mu_\alpha\delta^\nu_\beta - g_{\alpha\beta}g^{\kappa\mu}g^{\lambda\nu}\partial_\kappa\phi\partial_\lambda\phi) + 2g_{\alpha\beta}g^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi) =$$

$$= -\frac{1}{2}\int d^4x\sqrt{-g}((G^{\alpha\beta} - T^{\alpha\beta})P\delta\tilde{g}_{\mu\nu}(\delta^\mu_\alpha\delta^\nu_\beta - g_{\alpha\beta}g^{\kappa\mu}g^{\lambda\nu}\partial_\kappa\phi\partial_\lambda\phi) + (G^{\alpha\beta} - T^{\alpha\beta})2g_{\alpha\beta}g^{\kappa\lambda}\partial_\kappa\delta\phi\partial_\lambda\phi))$$

$$= -\frac{1}{2}\int d^4x\sqrt{-g}(P\delta\tilde{g}_{\mu\nu}(((G^{\mu\nu} - T^{\mu\nu}) - (G - T)g^{\kappa\mu}g^{\lambda\nu}\partial_\kappa\phi\partial_\lambda\phi) + 2(G - T)g^{\kappa\lambda}\delta\partial_\kappa\phi\partial_\lambda\phi))$$

where G and T the trace (contraction with $g_{\mu\nu}$) of the Einstein tensor and the stress-energy tensor respectively. Treating now ϕ and $\tilde{g}^{\mu\nu}$ as independent variables, there are two independent equations of motion:

$$(G^{\mu\nu} - T^{\mu\nu}) - (G - T)g^{\kappa\mu}g^{\lambda\nu}\partial_\kappa\phi\partial_\lambda\phi = 0$$

$$\partial_\kappa(\sqrt{-g}(G - T)g^{\kappa\lambda}\partial_\lambda\phi) = 0$$

Dividing the last equation by $\sqrt{-g}$ this is written more conveniently

$$\frac{1}{\sqrt{-g}}\partial_\kappa(\sqrt{-g}(G - T)g^{\kappa\lambda}\partial_\lambda\phi) = \nabla_k((G - T)\partial_\lambda\phi) = 0$$

Taking the trace of the first one

$$(G - T) - (G - T)g^{\kappa\mu}g^{\lambda\nu}\partial_\kappa\phi\partial_\lambda\phi g_{\mu\nu} = (G - T)(1 - g^\kappa_\nu g^{\lambda\nu}\partial_\kappa\phi\partial_\lambda\phi) =$$

$$= (G - T)(1 - g^{\kappa\lambda}\partial_\kappa\phi\partial_\lambda\phi) = 0$$

that can be satisfied also for $G - T \neq 0$ and this repoduces the identity $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1$ (that follows directly from $g^{\mu\nu} = \frac{1}{P}\tilde{g}^{\mu\nu} \rightarrow g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi = 1$). And even for $T^{\mu\nu} = 0$ the equations for the gravitational field have nontrivial solutions for the conformal mode. Thus the gravitational field acquires an extra degree of freedom shared by the scalar field ϕ and a conformal factor of the physical metric ($g_{\mu\nu} = P\tilde{g}_{\mu\nu}$).

The resulting equation of motion is:

$$G^{\mu\nu} = T^{\mu\nu} + \tilde{T}^{\mu\nu}$$

where $\tilde{T}^{\mu\nu} = (G - T)g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\phi\partial_\beta\phi$. Comparing this to the stress-energy tensor of a prefect fluid $T^{\mu\nu}_{per} = (\epsilon + p)u^\mu u^\nu - pg^{\mu\nu}$, where ϵ is the enrgy density, p is the pressure and u^μ is the four velocity of the fluid that satisfies (for the perfect fluid) the normalization condition $u^2 = u^\mu u_\mu = 1$. Setting $p = 0$ it is natural the identification between $T^{\mu\nu}_{per}$ and $\tilde{T}^{\mu\nu}$ by:

$$\epsilon = (G - T), u^\mu = g^{\mu\nu}\partial_\nu\phi$$

, so the extra degree of freedom, also for $T = 0$, has non zero energy density, imitating a dust perfect fluid. As for the stress-energy tensor for a perfect fluid this one satisfies a conservation law, that is $\nabla_\mu\tilde{T}^\mu{}_\nu = \nabla_\mu\tilde{T}^{\mu\alpha}g_{\alpha\nu} = 0$

$$\tilde{T}^{\mu\alpha}g_{\alpha\nu} = (G-T)g^{\mu\delta}g^{\alpha\beta}\partial_\delta\phi\partial_\beta\phi g_{\alpha\nu} =$$

$$= (G-T)\partial^\mu\phi g^\beta_\nu\partial_\beta\phi = (G-T)\partial^\mu\phi\partial_\nu\phi$$

And so

$$\nabla_\mu\tilde{T}^\mu_\nu = \nabla_\mu((G-T)\partial^\mu\phi\partial_\nu\phi) =$$

$$= \partial_\nu\phi\nabla_\mu((G-T)\partial^\mu\phi) + (G-T)\partial^\mu\phi(\partial_\nu\phi) =$$

$$= \partial_\nu\phi\nabla_\mu((G-T)\partial^\mu\phi) = 0$$

for the equation of motion $\nabla_k((G-T)\partial_\lambda\phi) = 0$ and because by differentiating $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1$, $\partial^\mu\nabla_\mu\partial_\nu\phi = 0$ and $\nabla_\nu\partial_\mu\phi = \nabla_\mu\partial_\nu\phi$.

After completed the presentation of mimetic dark matter, it's time to study a particular case of it.

4.3 Mimetic Dark Matter in Robertson-Walker Metric

Writing these equations in the Robertson-Walker metric in a flat geometry($k = 0$) (note that the space-time is not flat) with a comoving observer,

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j$$

Considering the conformal transformation $dt \to a(t(\eta))d\eta$,

$$ds^2 = a^2(\eta)(d\eta^2 - \delta_{ij}dx^i dx^j)$$

, $g_{\mu\nu} = a^2(\eta)diag(1,-1,-1,-1) = a^2(\eta)\eta_{\mu\nu}$. The aim is to find equations for the scale function $a(\eta)$.

The formulation of the problem is equivalent to [6]

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}[R(g_{\mu\nu}) + L_m - \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1)]$$

with the constraint $g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1$ and λ is a Lagrange multiplier that plays the role of the energy density.

In this case: $g = \det(g_{\mu\nu}) = -a^8(\eta)$, $R = -6\frac{a''}{a^3}$.

Then,

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}[R(g_{\mu\nu}) - \lambda(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 1) + L_m]$$

Considering the fact that ϕ depends only on $x^0 = \eta$ (see 3.1),

$$S = -\frac{1}{2}\int d^4x a^4[-6\frac{a''}{a^3} - \lambda(a^{-2}\eta^{00}\partial_0\phi\partial_0\phi - 1) + L_m] =$$

$$= -\frac{1}{2}\lim_{V\to\infty} V\int d\eta[-6\frac{a''}{a^3} - \lambda(a^{-2}\phi'^2 - 1) + L_m]a^4 = S_g + S_\lambda + S_m$$

The integration on space and time were separated. The integration on space is over an infinite volume. Considering now the class of lagrangian densities:

$$L_V = -\frac{1}{2}V[-6\frac{a''}{a^3} - \lambda(a^{-2}\phi'^2 - 1) + L_m]a^4$$

, for each $V \neq 0$ there are the same equations of motion (they differ only by a non zero factor). For simplicity L_1 is considered. Then the action become (if $L_m = 0$)

$$S = -\frac{1}{2}\int d\eta[-6\frac{a''}{a^3} - \lambda(a^{-2}\phi'^2 - 1)]a^4 = -\frac{1}{2}\int d\eta[-6aa'' - \lambda a^4(a^{-2}\phi'^2 - 1)]$$

Without considering the surface elements the action S becomes:

$$S = -\frac{1}{2}\int d\eta[6a'^2 - \lambda a^4(a^{-2}\phi'^2 - 1)]$$

Variation of the action with respect to a gives:

$$\delta_a S = \delta_a S_g + \delta_a S_\lambda = -\frac{1}{2}\delta_a\int d\eta 6a'^2 + \frac{1}{2}\delta_a\int d\eta a^4\lambda(a^{-2}\phi'^2 - 1) =$$

$$= -\frac{1}{2}\int d\eta 6(-2a'')\delta a + \frac{1}{2}\int d\eta[\frac{\partial(\lambda a^4)}{\partial a}(a^{-2}\phi'^2 - 1)\delta a + a^4\lambda\delta_a(a^{-2}\phi'^2)] =$$

$$= 6\int d\eta a^{''}\delta a + 1\int d\eta[\frac{\partial(\lambda a^4)}{\partial a}(a^{-2}\phi'^2 - 1) - a^4\lambda\phi'^2\frac{2}{a^3}] =$$

$$= 6\int d\eta a^{''}\delta a + \frac{1}{2}\int d\eta[\frac{\partial(\lambda a^4)}{\partial a}(a^{-2}\phi'^2 - 1) - a\lambda\phi'^2 2]\delta a = 0 \;\; \forall\delta a$$

Then

$$6a'' + \frac{1}{2}[\frac{\partial(\lambda a^4)}{\partial a}(a^{-2}\phi'^2 - 1) - a\lambda\phi'^2 2] = 0$$

Variation of the action with respect to λ gives:

$$\frac{\phi^{'2}}{a^2} - 1 = 0 \rightarrow \phi'^2 = a^2$$

Variation of the action with respect to ϕ gives:

$$\frac{d}{d\eta}(\phi'\lambda a^2) = 0$$

Using these results ($\phi' = a$):

$$\phi'\lambda a^2 = \lambda a^3 = C \rightarrow \lambda = \frac{C}{a^3}$$

where C is a real constant, this is gives a residual matter, whose energy density decreases with the expansion of the universe, as the dark matter does. With respect to the gravitational interaction this new mimetic dark matter behaves in the same way as the usual one, but it does not participate in any other interaction besides the graviational one.

Now

$$6a'' - a\lambda\phi'^2 = 0$$

Or

$$6a^{''} - C = 0$$

Finally

$$a'' = \frac{C}{6}$$

4.4 Background cosmological solutions

In the case of Robertson-Walker metric and in absence of matter the equation to study is one as for a point particle (say of mass m) in classical physics (1D dimension), under an acceleration given by:

$$a''(\eta) = \frac{C}{6}$$

The velocity of the point particle is

$$a'(\eta) = \frac{C}{6}\eta + B$$

And the position at a given time

$$a(\eta) = \frac{C}{12}\eta^2 + B\eta + A \to a(\eta) = C\eta^2 + B\eta + A$$

with a redefinition of C (from the left to the right side of the second =), where A, B are real constants that depend only on spatial coordinates. $A = 0$, imposed by $a(\eta = 0) = 0$. $B > 0$ for the expansion of the universe from the beginning. And $C > 0$ because the scale factor can't be negative for each time(indeed it is used to measure proper distances that are positive by definition).

So $a(\eta) = C\eta^2 + B\eta = \eta(C\eta + B)$ is a background evolution of a perfect fluid universe with equation of state $p = 0$(for $p \neq 0$ see [4], and also [1]), that mimic cold dark matter. And the energy density is

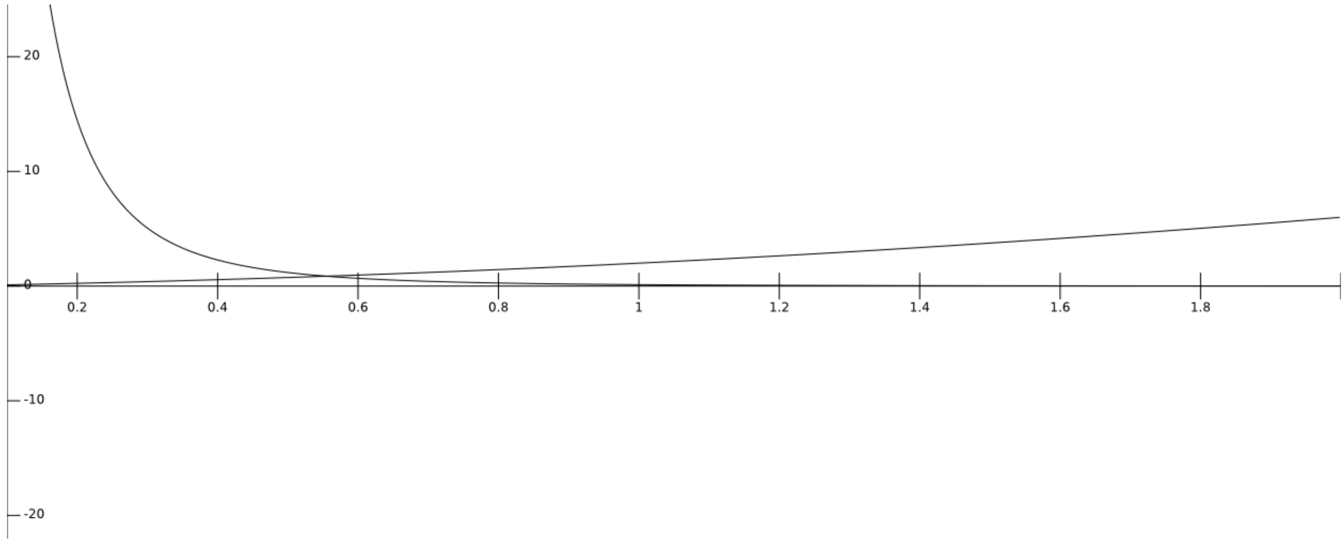
$$\lambda = \frac{C}{\eta^3(C\eta + B)^3}$$

In the image below is given an example of evolution for a particular case.

4.5 Conclusions

First, after a brief review of general relativity and cosmology it was analyzed the mimetic gravity theory, proposed as an explanation for dark matter. The resulting equations of motion present an extra term, that appears also without the canonical stress-energy tensor, that mimics the cold dark matter.

Figure 4.1: Evolution of scale factor and energy density for $C = B = 1$



Then it was made a simple study of a Universe, with Robertson-Walker metric, dominated by mimetic dark matter. The evolution is similar to that of a Universe dominated by non relativistic matter(setting $A = B = 0$ and $C = 1$). The elementary study made here showed that this simple model demonstrates the expansion of the Universe (an expansion that will last forever).

Possible simple generalizations of this thesis can be maked by introducing new sources of energy, such as ordinary matter or radiation or also adding a potential.

It also be interesting to find a way to derive mimetic gravity without introducing any extra scalar field λ (in this case the energy density of mimetic dark matter) in the action.

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